$$(2)=x^3-\tfrac{1}{2}ax^2-[y^2-\tfrac{1}{2}by+(bc^2/8y)]x+\tfrac{1}{8}ac^2=0,$$

$$\text{or } x^3-4x^2-[y^2-y+(25/y)]x+100=0 \ldots\ldots (4).$$

Eliminating x between (3) and (4) we get,

$$128102400y^{12}-536601600y^{11}-9985725784y^{10}+36190002752y^9$$
$$+30783923264y^8-100480598548y^7-4555231759005y^6$$
$$-4948451989304y^5+108549292200950y^4+26216813125200y^3$$
$$-54537984439125y^2+268623315000y-49234858937500=0 \ldots (5).$$

The prodigious amount of work necessary to arrive at (5) is wonderful, and great credit is due Mr. Bell for the above solution.

DIOPHANTINE ANALYSIS.

72. **Proposed by H. C. WILKES, Skull Run, W. Va.**

Given $x^2+y^2+z^2=p^2+q^2+r^2$, to find unequal integral values for x, y, z, p, q, and r.

I. **Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.**

The conditions of the problem are satisfied in the following six identities, in which m, n and r represent any integers, the values being so chosen as to avoid *zero* in the residual quantities,

$$(m+n+r)^2+(m-r)^2+(n-r)^2=(m+n-r)^2+(m+r)^2+(n+r)^2,$$
$$(m+n+r)^2+(m-n)^2+(n-r)^2=(m-n+r)^2+(m+n)^2+(n+r)^2,$$
$$(m+n+r)^2+(m-n)^2+(m-r)^2=(m-n-r)^2+(m+n)^2+(m+r)^2,$$
$$(m+n-r)^2+(m-n)^2+(m+r)^2=(m-n+r)^2+(m+n)^2+(m-r)^2,$$
$$(m+n-r)^2+(m-n)^2+(n+r)^2=(m-n-r)^2+(m+n)^2+(n-r)^2,$$
$$(m-n+r)^2+(m-r)^2+(n+r)^2=(m-n-r)^2+(m+r)^2+(n-r)^2.$$

These equations can be reduced to the following two formulas :

$$\left(\frac{p+2q+2r}{3}\right)^2+\left(\frac{2p+q-2r}{3}\right)^2+\left(\frac{2p-2q+r}{3}\right)^2=p^2+q^2+r^2 \ldots\ldots\ldots\ldots (1),$$

$$\left(\frac{2q-p+2r}{3}\right)^2+\left(\frac{2p+2q-r}{3}\right)^2+\left(\frac{2p-q+2r}{3}\right)^2=p^2+q^2+r^2 \ldots\ldots\ldots\ldots (2).$$

To insure integral results, assign to p, q, and r multiples of 3.

As 3 things can be arranged in 6 different ways, there may be made, in regard to p, q, and r, 6 different substitutions with each set of assigned values.

In Formula (1), these substitutions will produce 3 different sets of values

when the assigned values are different numbers. In Formula (2), however, only one new set will be obtained for each set of assigned values.

To obtain directly the 3 sets of values by Formula (1), arrange the set of assigned values in the order of their numerical greatness beginning with the largest number ; then substitute these respectively for p, q, r ; q, p, r ; r, q, p.

II. Solution by CHARLES C. CROSS, Libertytown, Md.

$$(m-n+p+q)^2+(m-p-q)^2+(m+n)^2=(m+n-p-q)^2+(m+p+q)^2+(m-n)^2,$$

$$(m+n-p+q)^2+(m-n-q)^2+(m+p)^2=(m-n+p-q)^2+(m+n+q)^2+(m-p)^2,$$

$$(m+n+p-q)^2+(m-n-p)^2+(m+q)^2=(m-n-p+q)^2+(m+n+p)^2+(m-q)^2.$$

Three sets of the sum of three squares equals the sum of three other squares, having no term in common.

Let $m=25$, $n=10$, $p=3$ and $q=1$.

Then $19^2+21^2+35^2=31^2+29^2+15^2$; $33^2+14^2+28^2=17^2+36^2+22^2$, and $37^2+12^2+26^2=13^2+38^2+24^2$.

Three other sets of the sum of three squares equal to the sum of three other squares are given below, but they each have two terms in common.

$$(m+n+p+q)^2+(m-n-p)^2+(m-q)^2=(m-n-p-q)^2+(m+n+p)^2+(m+q)^2,$$

$$(m+n+p+q)^2+(m-p-q)^2+(m-n)^2=(m-n-p-q)^2+(m+p+q)^2+(m+n)^2,$$

$$(m+n+p+q)^2+(m-n-q)^2+(m-p)^2=(m-n-p-q)^2+(m+n+q)^2+(m+p)^2.$$

If we let $m=am$, $n=bn$, $p=cp$, and $q=dq$, the above sets become very general.

III. Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

It is manifest that if we find more than one integral value for x, y, and z, in $x^2+y^2+z^2=a$ the question is solved.

Take $x^2+y^2+z^2=p^2$; assume $z^2=2xy$ and reducing we have $x+y=p$, and $y=p-x$, and $z^2=2px-2x^2=\square=$(say) q^2x^2. Then

$$x=\frac{2p}{q^2+2},\ y=\frac{q^2p}{q^2+2},\text{ and } z=\frac{qp}{q^2+2}$$

in which q may be any number. Taking $q=1$, 2, 3, etc., we obtain values of x, y, and z in terms of p, and any number of them that we choose. To obtain integral values, take p=the greatest common multiple of the denominators of the values taken, and we have as many values of x, y, and z, the sum of whose squares is constant ; and of course the sum of the squares of each set equals the sum of the squares of every other set. For example, take $q=1$, 2, 3, 4.

$$x=\frac{2p}{3},\quad \frac{2p}{6},\quad \frac{2p}{11},\quad \frac{2p}{18}.$$

$$y=\frac{p}{3},\quad \frac{4p}{6},\quad \frac{9p}{11},\quad \frac{16p}{18}.$$

$$z=\frac{2p}{3},\quad \frac{4p}{6},\quad \frac{6p}{11},\quad \frac{8p}{18}.$$

Take $p=198$.

$x=132$	66	36	22.
$y=66$	132	162	176.
$z=132$	132	108	88.

Rejecting the first or second values, we have three solutions of the question. Of course, we may take any two different sets of values and obtain one solution of the question.

IV. Solution by A. H. BELL, Hillsboro, Ill.

Let $x^2+y^2+z^2=p^2+q^2+r^2=(a^2+b^2+e^2)(c^2+d^2)$.

$(c^2+d^2)=\square$.

Then the general value will become $(ac\pm bd)^2+(bc\mp ad)^2+e^2(c^2+d^2)$.

By substitution we obtain 7 sets of values. When paired there are 12 sets of unequal values answering the requirements of this problem.

V. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Let $x=6m$, $y=10m$, $z=11m$, $p=7m$, $q=8m$, $r=12m$.

$\therefore\ x^2+y^2+z^2=p^2+q^2+r^2$.

$\therefore\ (6m)^2+(10m)^2+(11m)^2=(7m)^2+(8m)^2+(12m)^2=257m^2$, where m can have any integral value.

VI. Solution by the PROPOSER.

Let $x=m^2+n^2$, $y=m^2+mn-n^2$, $z=m^2-mn-n^2$.

We have $\overset{x^2}{(m^2+n^2)^2}+\overset{y^2}{(m^2+mn-n^2)^2}-\overset{z^2}{(m^2-mn-n^2)^2}=(m^2+2mn-n^2)^2=p^2$.

$$\overset{x^2}{(m^2+n^2)^2}+\overset{z^2}{(m^2-mn-n^2)^2}-\overset{y^2}{(m^2+mn-n^2)^2}=(m^2-2mn-n^2)^2=q^2.$$

$$\overset{z^2}{(m^2-mn+n^2)^2}+\overset{y^2}{(m^2+mn+n^2)}-\overset{x^2}{(m^2+n^2)^2}=(m^4-4m^2n^2+n^4)=z^2r^2.$$

Hence to find integral numbers that will fit the equation, we must find integral values for m and n that will make $(m^4-4m^2n^2+n^4)$ a square.

Let $m^4-4m^2n^2+n^4=r^2$ .... (1).

$m^4-4m^2n^2+4n^2=(r+a)^2$ .... (2).

$m^4-2m^2n^2+n^4=(r+b)^2$ .... (3).

From (1)+(2) we have $r=\dfrac{3n^4-a^4}{2a}$ ..... (4), $m^2=2n^2+\dfrac{3n^4+a^2}{2a}$ .... (5).

From (1)+(3) we have $r=\dfrac{2m^2n^2-b^2}{2b}$ ... (6), $m^2=n^2+\dfrac{2m^2n^2+b^2}{2b}$ ... (7).

From (4)+(6), $2am^2n^2-3bn^4=ab^2+a^2b$ .... (8).

From (5)+(7), $2am^2n^2-9bn^4-2abn^2=ab^2+a^2b$ ....(9).

Subtracting (9) from (8), $2abn^2=2ab(b-a)$. Whence $b-a=n^2$.

Factoring (8), $n^2(2am^2-3bn^2)=ab(b-a)$. Whence $2am^2-3bn^2=ab$.

Put $2a=n$ and by easy reduction we have $2m^2=b(6n+1)$. Since $\frac{1}{2}(6n+1)$ cannot be a square $b/2$ must be a square to make an integral. Then $m^2=b/2(6n+1)$ is $(n^2+\frac{1}{2}n)/2\times(6n+1)$. $n=4$ being the only value that will make both factors a square, $m=\sqrt{[(n^2+\frac{1}{2}n)/2]}\times\sqrt{(6n+1)}=15$.

$\therefore$ $x=241$, $y=269$, $z=149$, $p=329$, $q=89$, $r=191$.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

106. **Proposed by ELMER SCHUYLER, High Bridge. N. J.**

What is the amount of \$1000 at compound interest for 3 years at 6%, if it be compounded every instant?

107. **Proposed by R. V. ALLEN, Hooker Station, Ohio.**

A barn, $ABCD$, length $AB=b$ feet, width $AD=a$ feet, standing in an open field, has a horse tethered to a point, P, in the side, AB, distance $AP=c$ feet, with a rope R feet long. Over what area can the horse graze ?

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 10.

ALGEBRA.

94. **Proposed by J. W. YOUNG, Columbus, Ohio.**

Solve : $\left[\dfrac{x^2+14x+1}{p^4+14p^2+1}\right]^3=\dfrac{x(x-1)^4}{p^2(p^2-1)^4}$.

Burnside and Panton's *Theory of Equations*, page 148, ex. 17.

95. **Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.**

Substitute *numbers* in place of the letters in the following pattern : ...... $\triangle=\sqrt{81^2a^2b^2c^2}=81abc$..... b^2+c^2, a^2+c^2, a^2+b^2; and compute the areas and sides of the whole nest of integral, rational triangles.

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 10.

GEOMETRY.

114. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.**

If a variable ellipse hyperosculate a fixed ellipse at the extremity of the minor axis, the locus of the foci is a circle whose diameter is equal to the radius of curvature.